Final synopsis of Ph.D. thesis entitled

A Study of Fractional Differential Equations and Its Applications

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Research Guide

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Final Synopsis

Title: A Study of Fractional Differential Equations and its Applications.

Introduction:

Fractional differential equations (FDEs) generalize classical calculus by allowing derivatives of arbitrary order, making them ideal for modeling systems with memory and hereditary properties. This capability makes FDEs particularly effective in capturing long-range temporal or spatial dependencies across diverse fields such as engineering, physics, biology, and finance. Foundational work by Podlubny [1] laid the theoretical groundwork, followed by Kilbas et al. [2], who expanded its applications across scientific domains. Lakshmikantham et al. [3] explored existence and uniqueness results in abstract spaces, emphasizing the role of Banach spaces, while Diethelm [4] provided a comprehensive overview of both theory and numerical methods.

Analytical methods like the Power Series Method (PSM), Variational Iteration Method (VIM), and Adomian Decomposition Method (ADM) have been successfully applied to solve FDEs. Jafari and Seifi [5] demonstrated the versatility of ADM for linear and nonlinear problems, while Odibat and Momani [6] highlighted the computational simplicity of VIM. The development of fractional derivatives with non-singular kernels such as those proposed by Caputo and Fabrizio [7] and Atangana and Baleanu [8] further enhanced convergence and physical interpretation of memory effects in dynamic systems. These advances have driven interest in hybrid methods that integrate traditional and modern approaches for practical problem-solving.

Recently, Caputo-Fabrizio derivatives used in fixed transform domains (FTD) for groundwater modelling which offers a novel space-time framework [9]. Further, Said et al. [10] demonstrated the effectiveness of fractional-order low-pass filters in signal processing applications. However, despite these advances, existing solution techniques often lack generality particularly within the Banach algebra setting and the adoption of modern fractional derivatives like Caputo-Fabrizio and Atangana–Baleanu remains limited. The need for robust, hybrid analytical techniques (e.g., PSM,ADM,VIM,LTM) tailored to complex real-world systems continues to grow, especially in fields requiring accurate memory-based modelling.

Our research focuses on developing analytical and hybrid solution methods for FDEs and applying them to practical domains like groundwater modeling and signal filtering. Particularly, the generalized flow in class of partial differential equation and generalized initial value problem in low-pass signal filters under fractional-order set up.

Objectives:

- To study the existence and uniqueness of solution for FDEs using Banach and Schauder fixed point theorems.
- To study and implement a hybrid solution approach combining power series, Adomian decomposition, Variational iteration and Laplace transform methods.
- To apply Caputo-Fabrizio fractional differentiation in solving real-world problems like groundwater modeling and signal processing.
- To analyze and compare the accuracy and efficiency of various analytical methods.

This research addresses the need for generalized and hybrid analytical methods for solving FDEs involving Caputo-Fabrizio and Atangana-Baleanu derivatives. It also explores the existence and uniqueness of solutions using fixed point theory in Banach algebras and applies to signal processing and groundwater flow modelling.

The thesis shall comprises of following five chapters. Chapter I will brief about state of the art development of the topic and would help identifying the research gap. The prerequisites for subsequent chapters, such as definitions, properties and remarks are enlisted for development of the thesis. Chapter II deals with the existence and uniqueness results in the Banach Algebras, employing fixed point theory. The variety of analytical and numerical methods used in Chapter III to explore the nonlinear fractional-order initial value problem. The generalized flow equation for groundwater modelling application is explored with the help of fixed transform domain method with Caputo-Fabrizio derivative in Chapter IV. The last Chapter is on the fractional low-pass signal problem with fractional calculus sense.

Chapter 1: Introduction

This chapter presents a comprehensive introduction to FDEs, highlighting their historical development, mathematical foundations, and increasing relevance across diverse scientific and engineering disciplines. The chapter outlines the evolution of fractional calculus from its classical roots to modern applications and emphasizes the key motivations behind studying FDEs. Definitions of fractional derivatives were discussed including the Riemann–Liouville, Caputo, and Grünwald–Letnikov, Hadamard, Marchaud, Weyl , Conformable , Prabhakar derivative. Their mathematical properties and comparative advantages are given. Additionally, the chapter provides an extensive literature review and this chapter identifies the existing research gaps. The chapter concludes by summarizing the structure of the thesis and outlining the content of each chapter.

Chapter 2: Existence, Uniqueness of Solution to Fractional Differential Problem in Banach Algebra

We considered the fractional-order initial value problem:

$$D^{\alpha}x(t) = f\big(t, x(t)\big), \quad x(0) = x_0, \quad 0 < \alpha < 1.$$

We applied two fundamental fixed-point theorems - Banach and Schauder fixed point theorem to study the existence and uniqueness of solution.

Chapter 3: A Hybrid Approach for Solving Fractional Differential Equations

In this chapter, we applied the hybrid method combining the Power Series Method (PSM)**,** Adomian Decomposition Method (ADM)**,** Variational Iteration Method (VIM) and Laplace Transform Method (LTM) to solve fractional differential equations involving Caputo-Fabrizio (CF) and Atangana-Baleanu (AB) derivatives. Considered One-Term Fractional Initial Value Problem:

$$ {}_0^C D_t^{\frac{1}{2}} y(t) = cost, \quad y(0) = 0\ , \ \ t > 0.$$

After solving the equations using each method, we compared their results through graphical analysis.

Chapter 4: Space-Time Caputo-Fabrizio Fractional Differentiation Through a Fixed Transform-Based Domain Approach for Groundwater Model.

This chapter presents a novel fractional modelling approach for groundwater flow using space-time fractional differential equations involving the Caputo-Fabrizio (CF) derivative. A new Fixed Transform-Based Domain (FTD) methodology is introduced to simplify and effectively solve these equations, preserving memory effects and non-local behaviour inherent in subsurface flow dynamics.

The model is developed by combining Darcy's law and the principle of mass conservation, and incorporates Caputo-Fabrizio time-fractional derivatives to better represent anomalous diffusion. The governing equation for groundwater flow with a time-fractional component is given by:

$$^{CF}D_t^{\alpha}\phi(r,t) = \frac{K}{S}\nabla^2\phi(r,t),$$

where,

- $^{CF}D_t^{\alpha}$ denotes the Caputo-Fabrizio time-fractional derivative of order $\alpha \in (0,1]$
- $\phi(r,t)$ piezometric head as a function of radial distance and time,
- K: hydraulic conductivity,
- S: specific storage,
- ∇^2: Laplacian operator in radial coordinates.

The initial and boundary conditions are given by

Initial condition: $\phi(r,0) = ar$

Boundary condition: $\frac{\partial\phi(r,t)}{\partial r}|_{r=0} = -\frac{Q}{2\pi Kt}$

The graphical outputs demonstrate how fractional-order modelling with the CF derivative better captures the nonlinear and anomalous characteristics of groundwater flow, particularly in heterogeneous porous media.

Chapter 5: Fractional Calculus Approaches in Signal Processing: A Focus on Filtering Techniques

We demonstrate how a Fractional Order Low-Pass Filter (FOLPF) works in the time domain, we need to apply the filter to an signal constituents. A low-pass filter allows signals with frequencies lower than a certain cutoff frequency to pass through and attenuates signals with frequencies higher than the cutoff.

We consider an input signal consisting of a mixture of low and high-frequency components. This can be represented as:

$$x(t) = sin(2\pi 10t) + 0.5sin(2\pi 100t)$$

where:

- The first term $sin(2\pi 10t)$, represents a low-frequency component (10 Hz).
- The second term, $0.5sin(2\pi 100t)$represents a high-frequency component (100 Hz).

The applications discussed in Chapters IV and V shall lay the foundations for fractional order complexity through mathematical analysis. The thesis has wide applicability in the sense it starts with existence and uniqueness results and concludes, with an open to future research scope, some real-world theoretical applications.

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1. I. Podlubny, Fractional Differential Equations, *Academic Press*, 1999.
2. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, *Elsevier*, 2006.
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8. A. Atangana and D. Baleanu, 'New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model,' *Thermal Science*, vol. 20, no. 2, pp. 763–769, 2016.
9. M. L. Mahantane, 'A new method for modelling groundwater flow problems: Fractional differential equations approach,*' Master's thesis, University of the Free State*, 2019.
10. M. A. Alghamdi, 'Numerical approximation of the space-time Caputo-Fabrizio fractional diffusion equation, *Advances in Continuous and Discrete Models*, vol. 2016, no. 1, pp. 1–14, 2016.

List of Publications/Conferences Attended:

1. Published a Patent on "An Analytical approach for solving Fractional Differential Equations with Non-Homogeneous Boundary value issues" on 05/05/2023, in *Intellectual Property, India*. (Application No. 202341022909).
2. Published a research article in "Fractional Order Differential Equation in Banach Algebra" in *Turkish Journal of Computer and Mathematics Education* (2023).(Peer Reviewed)
3. Published a research article "Generalized Method for Solving Fractional Differential Equations" in *Journal of Xidian University*, Volume 18, Issue 12, December 24 Doi.10.37896/jxu18.12/005 ,ISSN.No.1001-2400, (Scopus).
4. Published a research article, "Space-Time Caputo Fabrizio Fractional Differentiation Through A Proposed Fixed Transform-Based Domain (FTD) Approach For Groundwater Model 'in *Comunications on Applied and Nonlinear Analysis'*,Vol 32 9s(2025)(Scopus).
5. Presented a paper in '6th International Conference on Emerging Trends on Sustainable Approaches in Science &Technology' on titled 'Generalized Method for Solving Fractional Differential Equations', 24-25 Feb2024.

6. Presented a paper in International Conference on Advancements in Humanities, Management, Science and Technology (ICAHMST) 2024 on titled 'Application of Fractional differential equation: A Review',24 th& 25 th July 2024.
7. Presented a paper in National Conference on Nonlinear Analysis and Applications (NCNAA-25) on titled 'Fractional Calculus Approaches in Signal Processing: A Focus on Filtering Techniques', during 30-31 January 2025.

Additionally Attended Conferences, Workshops,Training program

1. Attended 'National Conference on Mathematical Analysis and Applications – 2022'organized by the Department of Mathematics, Shivaji University, Kolhapur during 21-22 March, 2022.
2. Attended National Conference on Envisioning Mathematics Education in Line with NEP 2020 (21 Dec.2023) by Mahatma Gandhi Mahavidyalaya Ahmedpur, Dist. Latur (Maharashtra).
3. Attended a National Webinar on 'Introduction to Fractional Calculus' by Dr. Gunvant A. Biradar, organized by the department of Mathematics,Maitreyi College on September 29,2021.
4. Paticipated in IP Awareness Training program under National Intellectual Property Awareness Mission on June 03,2022, Organized by Intellectual Property Office,India in MGM University,Chhatrapati Sambhajinagar.
5. Participated in the Five Days International Level Online Workshop on Systematic Review of Literature using VOS viewer with Publish or Perish from 5.2.2024 to 9.2.2024.by GOMSGURU ACADEMY,Bangalore-560 064 .
6. Attended Online International Short-Term Training Program (STTP) on ' Fundamentals of MATLAB Programming and Applications' from 25.03.2024 to 29.03.2024 at Research and Publication Division,Scholars Gate-Research Training Institute, Tamil Nadu,India.
7. Attended Five Days Online International Workshop on 'How to write a Thesis' from 3.6.20247.6.2024,by GOMSGURU ACADEMY, Bangalore-560 064 .
8. Attended Five Days National Level Intensive Workshop cum Hands-on Training (Online) on 'LaTeX for Writing Research Papers & Scientific Documents' from 24.06.2024 to 28.06.2024,by A.C.T. Academy, Kerela Chapter.

9. Attended One day National Level Free e-workshop on ‘Introduction to Latex: A powerful Tools for Scientific Writing’on 04.03.2025, by A.C.T. Academy, Kerela Chapter.
10. Attended Mathwork webinar on “MATLAB Without Borders: Connecting your Projects with Python and other Open-Source Tools”27 February 2025,by Mathworks.
11. Attendend One day National Level research workshop on “Techniques to publish Research Papers in Peer-Reviewed Journals” on 06.12.2024,organized by A.C.T.Academy,Kerala.
12. Attended Mathwork webinar on ‘Teaching Intelligent Control Systems with MATLAB and Simulink’30 January 2025,by Mathworks.
13. Attended One day National Level research workshop on online Titled on ‘Geogebra Math Tool For 3D Visualization’on 10.12.2024 organized by A.C.T.Academy,Kerala.
14. Attended MATLAB EXPO 2024 on November 13-14,2024 online

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